ON THE GASTALDI – D'URSO FUZZY LINEAR REGRESSION

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Abstract
In the crisp regression models, the differences between observed values and calculates ones are suspected to be caused by random distributed errors, although these are due to observation errors and an unappropriate model structure.
So, the fuzzy character of model prevails.
The Fuzzy linear regression models (FLRM) are, roughly speaking, of two kinds:
Fuzzy linear programming (FLP) based methods and Fuzzy least squares (FLS) methods.
The FLP methods have been initiated by H.Tanaka (1982) and developed by H. Ishibuchi et al. The classical FLR model,

$$\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}_1\mathbf{X}_1 + \ldots + \mathbf{A}_k\mathbf{X}_k$$

has a explained Fuzzy triangular variable, Y, Fuzzy triangular coefficients {Aj} and crisp explanatory variables {Xj}: the parameters {Aj} of the model are estimated by minimizing the total indetermination of the model, so each data point lies within the limits of the response variable.

In a large number of situations the prediction interval of the FLR model were much less than the interval obtained applying classical the Multiple linear regression model (see V.M. Kandala – 2002, 2003).

However, this approach is somehow heuristic; on the other side, the LP model complexity overmuch increases as the number of data points increases.

The FLS approach (P. Diamond; Miin-Shen Yang, Hsien-Hsiung Liu – 1988 et al) is an extension of the classical OLS method, using various metrics defined on the space of the fuzzy numbers.

A significant number of recent works (McCauley- Bell (1999), J. deA. Sanchez and A. T. Gomez (2003) who used FLS to estimate the term structure of interest rates) deals with models with a fuzzy output, fuzzy coefficients and a crisp input vector.

All the fuzzy components are symmetric triangular fuzzy numbers: the main idea of the method is to minimize the total support of the fuzzy coeficients. Sometimes, different restrictions occur.
In our paper, we intend to build some examples for the P. d'Urso and T. Gastaldi models, that allow a comparative study on various options.
(Pierpaolo d'Urso & Tommaso Gastaldi in: A least square approach to fuzzy linear regression, Comp. Stat. & Data Analysis 2000)

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Introduction

The basic notion in the fuzzy theory is the fuzzy set or fuzzy number: in everyday mathematics, the corresponding notion is that of a set (or: crisp set). For a crisp set A, every element of the universe belongs or not to A, while for a fuzzy set A, every element of the subsequent universe has a degree of appurtenance to A, say, a number in the unit interval [0;1].

In this context, if an element **x** has the degree of appurtenance equal to 0, it "don't" belongs to A; if the degree of appurtenance equals to 1, then "sure" that **x** is an element of A; for the intermediate degrees of appurtenance e, say $\mu(\mathbf{x}) = 0{,}7$ the appurtenance is 70% possible and 30% not possible (for this reason, the fuzzy theory interferes with the so – known "possibility" theory).

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For a crisp set A, belonging to an universe X, the characteristic function $\mu_A(x): X \to \{0;1\}$ is given by

$$\mu_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

In the fuzzy case, the characteristic function, dubbed as *the membership function* is simply every function $\mu_A(x): X \to [0;1]$.

However, a series of special functions are routinely used, as in the following:

• **the triangular fuzzy numbers (Zimmermann)**: he's considering two continuous, decreasing shape functions, $\lambda, \rho: R^{+} \to [0,1]$ with $\lambda(1)=0; \lambda(0)=1; \rho(1)=0; \rho(0)=1$; then, a triangular fuzzy number, denoted by $(m, a, b)_{\lambda\rho}$ has the membership function A(x) given by

$$A(x) = \begin{cases} \lambda\left(\dfrac{m-x}{a}\right), \text{if } x \le m \\ \rho\left(\dfrac{x-m}{b}\right), \text{if } x > m \end{cases}$$

Here, **m** is called the center of $(m, a, b)_{\lambda\rho}$, **a** is the left spread and **b** is the right spread of $(m, a, b)_{\lambda\rho}$. If **a** = **b**, then $(m, a, b)_{\lambda\rho}$ is called a ***symmetrical triangular fuzzy number*** that will be denoted simply by $(m, a)_{\lambda\rho}$.

In most applications, the shape functions **ρ, λ** are supposed to be $\rho(x) = x; \lambda(x) = x$ and consequently, the membership function becomes

$$A(x) = \begin{cases} 1-\left(\dfrac{m-x}{a}\right), \text{if } m-a \le x \le m \\ 1-\left(\dfrac{x-m}{b}\right), \text{if } m < x \le m+b \end{cases}$$

As an alternative, for an ***exponential fuzzy number***, the membership function has the shape below:

$$A(x) = \begin{cases} \exp\left[-\left(\dfrac{m-x}{\sigma}\right)^{k}\right], \text{for } x \le m \\ \exp\left[-\left(\dfrac{x-m}{\sigma}\right)^{k}\right], \text{for } x > m \end{cases}$$

Here, the spread is given by the positive $\sigma > 0$.

- **the trapezoidal fuzzy number** (D. Ralescu; L. A. Zadeh; Dubois), denoted by (a, b, α, β) has the membership function **A(x) :R → [0;1]** given by

$$A(x) = \begin{cases} \dfrac{x - a + \alpha}{\alpha}, & a - \alpha \le x \le a \\ 1, & a < x < b \\ \dfrac{b + \beta - x}{\beta}, & b \le x \le b + \beta \\ 0, & \text{otherwise} \end{cases}$$

- For the symmetrical triangular fuzzy numbers (TFN), an **Euclidean distance** is available: let $A = (m_1, a_1)_{\lambda\rho}$, $B = (m_2, a_2)_{\lambda\rho}$ be TFN: then, the Euclidean distance between A, B is

$$d_{AB} = \sqrt{(m_1 - m_2)^2 + (a_1 - a_2)^2}.$$

If A, B are non-symmetrical, $A = (m_1, a_1, b_1)_{\lambda\rho}$, $B = (m_2, a_2, b_2)_{\lambda\rho}$ then, for $p_1, p_2, p_3 > 0$ suitable defined weights, $p_1 + p_2 + p_3 = 1$, the distance d_{AB} can be compute by using the next formula:

$$d_{AB} = \sqrt{p_1 \cdot (m_1 - m_2)^2 + p_2 \cdot (a_1 - a_2)^2 + p_3 \cdot (b_1 - b_2)^2}$$

PAPER CONTENT:

Fuzzy linear regression

Let's take into account the Linear Fuzzy regression model Y = $(a_0, r_0) + (a_1, r_1)$X, with:

- X – the explanatory variable, Y – the response variable
- (a_0, r_0), (a_1, r_1) the triangular symmetric fuzzy coefficients to be estimated
- X is supposed to be a crisp variable, Y = (c, s) a TSFN variable.

Remember that a triangular symmetric fuzzy number (TSFN), denoted by x = (a , r), has the membership function A(x): R → [0;1] given by

$$A(x) = \begin{cases} \dfrac{x - (a - r)}{r}, & \text{if } a - r \le x < a \\ \dfrac{(a + r) - x}{r}, & \text{if } a \le x \le a + r \\ 0, & \text{otherwise} \end{cases}$$

The spread of x is then equal to 2a: the centre of x is a.

The Tanaka approach, referred to as possibility regression, was to minimize the fuzziness of the model, represented by the total spread of the fuzzy coefficients. So, the method becomes an extension of the classical OLS method.

To illustrate this, let's consider the data in Table 1 below:

TABLE 1:

Y_i	X_i
$y_1 : (5 ; 2) = (c_1, s_1)$	$x_1 = 1$
$y_2 : (8 ; 3) = (c_2, s_2)$	$x_2 = 2$
$y_3 : (10 ; 2) = (c_3, s_3)$	$x_3 = 3$

With the linear regression model: $Y = (a_0, r_0) + (a_1, r_1)X$

The required approximations become

$$\begin{cases} (5;2) \approx (a_0, r_0) + (a_1, r_1) \\ (8;3) \approx (a_0, r_0) + 2 \cdot (a_1, r_1) \\ (10;2) \approx (a_0, r_0) + 3 \cdot (a_1, r_1) \end{cases}$$

From here the OLS conditions are derived

- for the centers: $\min\{(a_0 + a_1 - 5)^2 + (a_0 + 2a_1 - 8)^2 + (a_0 + 3a_1 - 10)^2\}$
- for the spreads: $\min\{(r_0 + r_1 - 2)^2 + (r_0 + 2r_1 - 3)^2 + (r_0 + 3r_1 - 2)^2\}$.

The solutions are: $\begin{cases} a_0 = 8/3 \\ a_1 = 5/2 \end{cases}; \begin{cases} r_0 = 7/3 \\ r_1 = 0 \end{cases}$, so the estimated model becomes

$$Y = (2,5;0) \cdot X + (2,67;2,33) \Leftrightarrow Y = 2,5 \cdot X + (2,67;2,33)$$

Having zero spread, the X coefficient is a crisp number.

This equation allow us to perform interpolations, for example

$x = 2,5 \rightarrow y = \frac{5}{2} \cdot 2,5 + (8/3;7/3) = (6,25;0) + (2,67;2,33) = (8,92: 2,33)$.

For a large number of explanatory variables, the matrix approach is more suitable. To illustrate this let's consider the data in TABLE 2.

TABLE 2:

sample	Y	X_1	X_2	…	X_p	the independent term
1	y_1	x_{11}	x_{12}	…	x_{1p}	1
2	y_2	x_{21}	x_{22}	…	x_{2p}	1
…	…	…	…	…	…	…
n	y_n	x_{n1}	x_{n2}	…	x_{np}	1

with: $\mathbf{y_1 = (c_1 , s_1); y_2 = (c_2 , s_2) ;…; y_n = (c_n , s_n)}$ **and** $X_1,\dots, X_n$ be crisp variables; the linear regression function will be

$$\mathbf{Y = (a_0 , r_0) + (a_1 , r_1)X_1 +(a_2 , r_2)X_2 + \dots + (a_p , r_p)X_p}$$

The matrix elements of the model will be the next:

• the n x (p + 1) data matrix $\mathbf{X}=\begin{pmatrix} \mathbf{x_{11}} & \mathbf{x_{12}} & \dots & \mathbf{x_{1p}} & \mathbf{1} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{x_{n1}} & \mathbf{x_{n2}} & \dots & \mathbf{x_{np}} & \mathbf{1} \end{pmatrix}$

• the components of the TSFN Y's data:

The n x 1 centers vector, $\mathbf{C}=\begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \dots \\ \mathbf{c_n} \end{pmatrix}$ and the n x1 spreads vector $\mathbf{S}=\begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \dots \\ \mathbf{s_n} \end{pmatrix}$

• the (p + 1) x 1 vectors of the TSFN's coefficients,

The centers , $\hat{\mathbf{a}}=\begin{pmatrix} \mathbf{a}_1 \\ \dots \\ \mathbf{a_p} \\ \mathbf{a}_0 \end{pmatrix}$ and the spreads $\hat{r}=\begin{pmatrix} r_1 \\ \dots \\ r_p \\ r_0 \end{pmatrix}$

The OLS method gives

$$\hat{\mathbf{a}}=(\mathbf{X'}\cdot\mathbf{X})^{-1}\cdot\mathbf{X'}\cdot\mathbf{C}\ ;\ \hat{\mathbf{r}}=(\mathbf{X'}\cdot\mathbf{X})^{-1}\cdot\mathbf{X'}\cdot\mathbf{S}$$

Let, for example, be the data in TABLE 3 below:

TABLE 3:

the sample	**Y**	**X_1**	**X_2**
1	**(3;2)**	**2**	**1**
2	**(5;1)**	**1**	**3**
3	**(7;3)**	**3**	**2**

We'll apply the regression model : $\mathbf{Y = (a_0, r_0) + (a_1, r_1)X_1 + (a_2, r_2)X_2}$
According to the previous notations and formula,

$$\mathbf{X} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix} ; \mathbf{C} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} ; \mathbf{S} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \Rightarrow \hat{\mathbf{a}} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} ; \hat{\mathbf{r}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

so, the estimated model will be : $\hat{\mathbf{Y}} = \mathbf{(-3, 0) + (2, 1)X_1 + (2, 0)X_2}$ **or**

$\hat{\mathbf{Y}} = \mathbf{-3 + (2, 1)X_1 + 2X_2}$ **.**

Here, the number of parameters being equal to the number of conditions, the estimated values $\hat{\mathbf{Y}}$ are equal to the observed ones,

$$x_1 = 2 ; x_2 = 1 \rightarrow \hat{y}_1 = (-3;0) + (4;2) + (2;0) = (3;2) \Leftrightarrow \hat{y}_1 = y_1$$

$$x_1 = 1 ; x_2 = 3 \rightarrow \hat{y}_2 = (-3;0) + (2;1) + (6;0) = (5;1) \Leftrightarrow \hat{y}_2 = y_2$$

$$x_1 = 3 ; x_2 = 2 \rightarrow \hat{y}_3 = (-3;0) + (6;3) + (4;0) = (7;3) \Leftrightarrow \hat{y}_3 = y_3 .$$

If there are more data than model parameters, a non-zero error can occur, as shown in the example below:

Let's consider the data in TABLE 4, subject to the model

$\mathbf{Y = (a_0, r_0) + (a_1, r_1)X_1 + (a_2, r_2)X_2}$

TABLE 4:

the sample	**Y**	**X_1**	**X_2**
1	**(4;3)**	**2**	**1**
2	**(7;2)**	**3**	**2**
3	**(9;5)**	**2**	**3**
4	**(8;2)**	**4**	**2**

The corresponding matrix elements will be

$$\mathbf{X}=\begin{pmatrix}2&1&1\\3&2&1\\2&3&1\\4&2&1\end{pmatrix};\mathbf{C}=\begin{pmatrix}4\\7\\9\\8\end{pmatrix};\mathbf{S}=\begin{pmatrix}3\\2\\5\\2\end{pmatrix}\Rightarrow$$

$$\Rightarrow\hat{\mathbf{a}}=(\mathbf{X'}\cdot\mathbf{X})^{-1}\cdot\mathbf{X'}\cdot\mathbf{C}=\begin{pmatrix}0{,}727\\2{,}5\\0\end{pmatrix};\hat{\mathbf{r}}=(\mathbf{X'}\cdot\mathbf{X})^{-1}\cdot\mathbf{X'}\cdot\mathbf{S}=\begin{pmatrix}-1{,}09\\1\\4\end{pmatrix}$$

The estimated regression function becomes $\hat{\mathbf{Y}}$ **= (0 ; 4) + (0,727 ; -1,09)**$\mathbf{X_1}$ **+ (2,5 ; 1)**$\mathbf{X_2}$**.**

Comparing the observed values with the estimated values, we obtain

$$x_1 = 2 ; x_2 = 1 \rightarrow \hat{y}_1 = (3{,}954;2{,}82) \Leftrightarrow \hat{y}_1 \approx y_1$$

$$x_1 = 3 ; x_2 = 2 \rightarrow \hat{y}_2 = (7{,}181;2{,}73) \Leftrightarrow \hat{y}_2 \approx y_2$$

$$x_1 = 2 ; x_2 = 3 \rightarrow \hat{y}_3 = (8{,}954;4{,}82) \Leftrightarrow \hat{y}_3 = y_3$$

$$x_1 = 4 ; x_2 = 2 \rightarrow \hat{y}_4 = (7{,}91;1{,}69) \Leftrightarrow \hat{y}_4 = y_4$$

A better approach is that of Tanaka’s symmetrical Doubly Linear Adaptive Fuzzy Regression Model, whose central idea is that the calculated spreads depend linearly to the calculated centers.

In this respect, for

$$\mathbf{a}=\begin{pmatrix}\mathbf{a_1}\\\ldots\\\mathbf{a_p}\\\mathbf{a_0}\end{pmatrix};\mathbf{C}=\begin{pmatrix}\mathbf{c_1}\\\mathbf{c_2}\\\ldots\\\mathbf{c_n}\end{pmatrix};\mathbf{S}=\begin{pmatrix}\mathbf{s_1}\\\mathbf{s_2}\\\ldots\\\mathbf{s_n}\end{pmatrix},\tilde{\mathbf{1}}=\begin{pmatrix}1\\1\\\ldots\\1\end{pmatrix}$$ - “n“ dimensional vector

We put

- the calculated centers: $\hat{\mathbf{C}}=\mathbf{X}\cdot\mathbf{a}$
- and, accordingly, the calculated spreads: $\hat{\mathbf{S}}=\hat{\mathbf{C}}\cdot\mathbf{b}+\tilde{\mathbf{1}}\cdot\mathbf{d}$

The optimum condition will be: **(minim)** $\left\|\mathbf{C}-\hat{\mathbf{C}}\right\|^2+\left\|\mathbf{S}-\hat{\mathbf{S}}\right\|^2$.

The next two ways to perform this:

• the "Doubly Linear" method, in which the vector **a** is estimated first using OLS method, so $\hat{a} = (X' \cdot X)^{-1} \cdot X' \cdot C$, the parameters of the Minimum model being the scalars b, d;

• the d'Urso and Gastaldi version, in which the unknown parameters are **a, b, d.**

As an example of applying the Doubly Linear model, let's consider the data in TABLE 5 below:

TABLE 5:

the sample	Y	X
1	(8 ; 3)	1
2	(5 ; 2)	2
3	(9 ; 4)	3

the corresponding model being the next

$(c_i ; s_i) = (a_0 ; r_0) + (a_1 ; r_1) x_i ; i = 1,2,3$

For

$$C = \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix}; S = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}; X = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}; a = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix},$$

There arrive that

$$\hat{C} = \begin{pmatrix} a_1 + a_0 \\ 2 \cdot a_1 + a_0 \\ 3 \cdot a_1 + a_0 \end{pmatrix}; \hat{C} - C = \begin{pmatrix} a_1 + a_0 - 8 \\ 2 \cdot a_1 + a_0 - 5 \\ 3 \cdot a_1 + a_0 - 9 \end{pmatrix};$$

$$\hat{S} = \begin{pmatrix} b \cdot (a_1 + a_0) + d \\ b \cdot (2 \cdot a_1 + a_0) + d \\ b \cdot (3 \cdot a_1 + a_0) + d \end{pmatrix}; \hat{S} - S = \begin{pmatrix} b \cdot (a_1 + a_0) + d - 3 \\ b \cdot (2 \cdot a_1 + a_0) + d - 2 \\ b \cdot (3 \cdot a_1 + a_0) + d - 4 \end{pmatrix}$$

Putting $\hat{a} = (X' \cdot X)^{-1} \cdot X' \cdot C = \begin{pmatrix} 0,5 \\ 6,33 \end{pmatrix}$ there obtains $a_1 = 0,5$; $a_0 = 6,33$ and consequently

$$\hat{C} = \begin{pmatrix} 6,83 \\ 7,33 \\ 7,83 \end{pmatrix}.$$

Finally, $\hat{S} - S = \begin{pmatrix} 6{,}83 \cdot b + d - 3 \\ 7{,}33 \cdot b + d - 2 \\ 7{,}83 \cdot b + d - 4 \end{pmatrix}$.

From the condition: **(minim)** $\left\| S - \hat{S} \right\|^2$ there results **b = 1** and **d = -4,33 .**

Having $\hat{S} - S = \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix}$, then $\hat{S} = \begin{pmatrix} 2{,}5 \\ 3 \\ 3{,}5 \end{pmatrix}$ and so:

- $\hat{y}_1 = (6{,}83; 2{,}5)$ versus $y_1 = (8; 3)$
- $\hat{y}_2 = (7{,}33; 3)$ versus $y_2 = (5; 2)$
- $\hat{y}_3 = (7{,}83; 3{,}5)$ versus $y_3 = (9; 4)$

For the Gastaldi – d'Urso variant, which not necessarily lead to better results, although the computations volume is incomparably greater, the parameters of the optimization problem

$$\textbf{(minim)}\ \left\| C - \hat{C} \right\|^2 + \left\| S - \hat{S} \right\|^2$$

Will be **a, b** and **d**.

According to the results of Gastaldi – d'Urso, the solutions to this problem are given by the equations below:

$$\hat{C} = X \cdot (X' \cdot X)^{-1} \cdot X' \cdot C \quad ; \hat{S} = X \cdot (X' \cdot X)^{-1} \cdot X' \cdot S$$

$$\overline{C} = \frac{1}{n} \cdot \tilde{1}' \cdot C \quad ; \overline{S} = \frac{1}{n} \cdot \tilde{1}' \cdot S$$

$$M_1 = C' \cdot \hat{S} - n \cdot \overline{C} \cdot \overline{S} \ ; M_2 = \|\hat{C}\|^2 - \|\hat{S}\|^2 + n \cdot \overline{S}^2 - n \cdot \overline{C}^2 \ ; M_3 = n \cdot \overline{C} \cdot \overline{S} - S' \cdot \hat{C}$$

The appropriate value of **b**, denoted by $\hat{\mathbf{b}}$, is derived from the equation

$$\mathbf{M_1 b^2 + M_2 b + M_3 = 0}$$

Therefore

$$\hat{\mathbf{d}} = \overline{\mathbf{S}} - \hat{\mathbf{b}} \cdot \overline{\mathbf{C}} \quad ; \quad \hat{\mathbf{a}} = \frac{(\mathbf{X'} \cdot \mathbf{X})^{-1} \cdot \mathbf{X'} \cdot (\mathbf{C} + \mathbf{S} \cdot \hat{\mathbf{b}} - \tilde{\mathbf{1}} \cdot \hat{\mathbf{b}} \cdot \hat{\mathbf{d}})}{\mathbf{1} + \hat{\mathbf{b}}^2}$$

For an already presented example, namely

TABLE 6:

sample	Y	X_1	X_2	X_3
1	(3; 1)	1	2	1
2	(6; 2)	3	1	2
3	(8; 2)	2	2	3
4	(7; 4)	1	4	3
5	(10; 3)	3	2	4

From the equation: $\mathbf{M_1 b^2 + M_2 b + M_3 = 0}$, we derive that $\hat{\mathbf{b}}_1 = \mathbf{0{,}309}$, $\hat{\mathbf{b}}_2 = -\mathbf{3{,}236}$ and finally:

$$\hat{\mathbf{b}}_1 = 0{,}309 \Rightarrow \hat{\mathbf{d}}_1 = 0{,}3 \;\; ; \; \hat{\mathbf{a}}_1 = \begin{pmatrix} 0{,}423 \\ 0{,}106 \\ 2{,}051 \\ 0{,}387 \end{pmatrix};$$

$$\hat{\mathbf{b}}_2 = -3{,}236 \Rightarrow \hat{\mathbf{d}}_2 = 224{,}4 \;\; ; \; \hat{\mathbf{a}}_2 = \begin{pmatrix} -0{,}423 \\ -0{,}541 \\ 0{,}297 \\ 8{,}063 \end{pmatrix}$$

By applying the relations:

$\mathbf{C} = \mathbf{X} \cdot \hat{\mathbf{a}} \;\; ; \mathbf{S} = \hat{\mathbf{C}} \cdot \hat{\mathbf{b}} + \tilde{\mathbf{1}} \cdot \hat{\mathbf{d}}$, we'll get

$$\mathbf{C}_1 = \begin{pmatrix} 389{,}7 \\ 392{,}5 \\ 394{,}2 \\ 394 \\ 396{,}66 \end{pmatrix} \; ; \; \mathbf{S}_1 = \begin{pmatrix} 120{,}71 \\ 121{,}58 \\ 122{,}11 \\ 122{,}05 \\ 122{,}88 \end{pmatrix}$$ - Obviously not convenient

$$\mathbf{C}_2 = \begin{pmatrix} 6{,}855 \\ 6{,}897 \\ 7{,}026 \\ 6{,}367 \\ 6{,}9 \end{pmatrix} \; ; \; \mathbf{S}_1 = \begin{pmatrix} 2{,}22 \\ 2{,}24 \\ 1{,}66 \\ 3{,}8 \\ 2{,}07 \end{pmatrix}$$ - This one being the feasible solution.

In the comparative TABLE 7 below, the next are presented:

- $\{\mathbf{y}_i^{\#}\}$ the values calculated using Gastaldi – d'Urso method
- $\{\hat{\mathbf{y}}_i\}$ the values calculated using the Doubly Linear method;
- $\{\tilde{\mathbf{y}}_i\}$ the values calculated using the OLS method:

TABLE 7:

$\mathbf{y}_i$	$\hat{\mathbf{y}}_i$	$\tilde{\mathbf{y}}_i$	$\mathbf{y}_i^{\#}$
(3; 1)	**(3,14 ; 1,06)**	**(3,14 ; 9,07)**	**(6,9 ; 2,2)**
(6; 2)	**(5,92 ; 1,95)**	**(5,92 ; 10,61)**	**(6,8 ; 2,24)**
(8; 2)	**(7,83 ; 1,9)**	**(7,83 ; 11,22)**	**(7,03 ; 1,66)**
(7; 4)	**(6,96 ; 3,96)**	**(6,96 ; 13,29)**	**(6,37 ; 3,8)**
(10; 3)	**(10,18 ; 3,07)**	**(10,18 ; 13,05)**	**(6,9 ; 2,07)**

Of course, not always the effectiveness of this methods is that in the Table 7: this effectiveness heavily depends on input data.

According to our information, there is no theorem stating the adequacy of each method.

Conclusion:

The basic idea of the paper is to analyze the influence of the spreads on the accuracy study.

In computing fuzzy regression – as well as other categories – there observe an automatically growing spreads with increasing number of fuzzy parameters.

Responsible for this shortcoming are the very definitions of the fuzzy operations itself. Major complications are arising when d'Urso - Gastaldi attempts to keep under control the spreads, by correlating these last with the other characteristics.

In our view, another approach consists in modifying the basic definitions as follow:

• the addition : $(a;\alpha)+(b;\beta)=(a+b;\max\{\alpha;\beta\})$

• the multiplication : $(a;\alpha)\cdot(b;\beta)=(a\cdot b;\max\{\alpha;\beta\})$

By adopting this viewpoints, we'll have the distribution,

$$(a;\alpha)\cdot[(b;\beta)+(c;\nu)]=(a;\alpha)\cdot(b;\beta)+(a;\alpha)\cdot(c;\nu)$$

thus this concept is a natural one.

On the other hand, is true that analytical optimization methods are difficult to carry out.

Our team is still trying to find solutions to these problems in occasion of studying some major economic applications.

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